SUMMING SERIES BY COMPUTER

KEN MEYER

1. Introduction

Throughout the latter half of the 19^{th} century and the early half of th 20^{th} century mathematicians investigated various methods to sum divergent series. The names Abel, Borel, Cesáro, et al. are associated with different summation methods. But starting in the last half of the 20^{th} century the electronic computer has given us a new, almost universal method of summing divergent series. The results of this new method are surprising and sometimes alarming.

2. Classical Methods

The standard method of summing a series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

is to investigate the limit of the sequence of partial sums $P_k = \sum_{n=0}^{k} a_n$, i.e., the sequence

$$P_0 = a_0,\ P_1 = a_0 + a_1,\ P_2 = a_0 + a_1 + a_2,\ \cdots .$$

The series is said to converge or diverge as does the sequence of partial sums. For example $\sum 2^{-n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ converges to 2, whereas the series $\sum(-1)^n = 1 - 1 + 1 - 1 + 1 + \cdots$ diverges since the partial sums alternate between 1 and 0..

The classical summation method of Cesáro investigates the sequences of averages of the partial sums,

$$A_N = \frac{1}{N+1}\sum_{k=0}^{N} P_k = \sum_{n=0}^{N}\left[\frac{N-n+1}{N+1}\right] a_n.$$

If A_N converges then the series is said to be Cesáro summable (or C-summable) and its Cesáro sum is $\lim_{N\to\infty} A_N$. If a series is convergent in the standard sense then it is C-summable to the same limit. But it is easy to see that the series $\sum(-1)^n = 1 - 1 + 1 - 1 + 1 + \cdots$ is now C-summable to 1/2. The sequence of partial sums P_k alternates between 1 and 0 so on average they get close to 1/2. So Cesáro sums a divergent series. See the classic book by Hardy [1] for more details on this and other classical summation methods. But now there is a new method!

3. The Baldwin Scientific Method

From time to time we are called upon to teach a mathematics course to students in the engineering building near our offices. This fine old structure is called Baldwin Hall. By listening to the conversations of our engineering colleagues in these halls, we have learned that they use the scientific method to develop a new and powerful summation procedure. We call this summation procedure the *Baldwin scientific method*. A series $\sum a_n$ is said to be convergent by the Baldwin scientific method (or be *BS-summable*) if the digits in the computed partial sums, $P_k = \sum_0^k a_n$, stabilize. That is, write a computer program to compute the partial sums and print them out. If after some time the displayed values do not change then the series is BS-summable and the last printed value is the BS-sum. If on ther other hand the computed values grow and the computer issues an error message like "overflow" then the series is *BS-divergent* to infinity.

In the mid 1980s I tested this method on several series using Basic on my TRS-80 computer, running at 2 kilohertz with 64 kilobytes of memory. To my surprise it was a very powerful method indeed. Now only did it sum the harmonic series, $\sum 1/n$ to 14.40369, but also showed that $\sum 1 = 1 + 1 + 1 + \cdots$, $\sum n = 1 + 2 + 3 + \cdots$, $\sum n^2 = 1 + 4 + 9 + \cdots$ are **BS-summable!**. However, not all series are BS-summable. The series $\sum n!$ is BS-divergent to infinity.

BS-Sums of Divergent Series

Series	BS-Sum	k for which $P_k = P_{k+1}$
$\sum (1/n)$	14.40369	$2,097,153$
$\sum 1$	1.677722×10^7	$16,777,216$
$\sum \ln n$	5.368709×10^8	$25,544,582$
$\sum n$	2.814750×10^{14}	$24,815,776$
$\sum n^2$	4.722366×10^{21}	$27,849,738$
$\sum e^n$	∞ DS-Divergent	90
$\sum n!$	∞ DS-Divergent	36

4. Theory

To understand Baldwin scientific method of summation one needs to know a little about a computer's internal representation of floating point numbers (sometimes incorrectly call reals). Real computers use binary representation, but we shall discuss a mythical computer called Puff which uses decimal representation to facilitate our understanding. In Puff a floating point number will be of the form

$$\pm d_1 d_2 d_3 \times 10^{\pm e}$$

where d_i and e are digits from 0 to 9 and except for the representation of 0 the lead digit d_1 is nonzero. (Imagine a decimal point before $d_1 d_2 d_3$.) All d_i and e are zero in the representation of 0. So for example 3.14 would be stored as $+314 \times 10^{+1}$. The $d_1 d_2 d_3$ is called the mantissa and the $\pm e$ the exponent. This is like scientific notation except that there are a fixed number of significant digits and a fixed exponent range. Thus in Puff floating point numbers lie between $.100^{-9}$ and $.999 \times 10^9$. If a computation leads to a smaller number then Puff issues an underflow message and if it leads to a larger number then Puff issues an overflow message. In Puff two such numbers are added, subtracted, multiplied and divided exactly and then the result is truncated to this range. Thus for example $102 \times 10^{+2} + 304 \times 10^{+1}$ is first computed to be $10.2 + 3.04 = 13.24$ and the truncated to $132 \times 10^{+2}$.

Let us see how Puff sums the series $\sum 1 = 1 + 1 + 1 + 1 + \cdots$. Some of Puff's partial sums are given in the table below.

k	P_k
1	$100 \times 10^{+1}$
9	$900 \times 10^{+1}$
100	$100 \times 10^{+3}$
999	$999 \times 10^{+3}$
1000	$100 \times 10^{+4}$
1001	$100 \times 10^{+4}$

Up to 999 no truncation occurs and the computed sum is the actual sum. At 1000 the last digit is dropped, but since it is 0 there is no loss. But 1001 is truncated to $100 \times 10^{+4}$ (=1000). Each new added 1 does not change Puff's representation of the sum. Thus on Puff the BS-sum of the series $\sum 1$ is 1000, i.e., the largest number held in the mantissa plus 1.

BS-convergence occurs when the next term to be added is small compared to the partial sum. This gives the *BS-ratio test* for convergence. On Puff as soon as $|a_{k+1}/P_k| < 10^{-4}$ then $P_k = P_{k+1}$ and the series is BS-convergent. However, if the partial sums get to large too fast then overflow occurs leading to *BS-divergence*.

5. Final Remarks

Actually the whole method is computer dependent. Thus, we do not have a single method, but a parameterized method of summation parameterized by computers. The table of sums given above was calculated on a very early home computer, a Radio Shack TRS 80 model III, using Basic. The reals on that machine had a mantissa of 24 bites and $2^{24} = 16,777,216$. Even at 2 kilohertz it did not take long to count to 16,777,216, but todays computers even though much faster need to count to numbers like

$$2^{79} = 604,462,909,807,314,587,353,088.$$

So $\sum 1$ will still be BS-summable on a modern computer, it will just take longer.

References

[1] G. D. Hardy, *Divergent Series*, Oxford Press, 1948.